

Problème 1

Afin d'exprimer le déplacement δ_C en C, on introduit une force auxiliaire fictive Q en C (Q prendra la valeur 0 par la suite)

1. Calcul des réactions en exprimant l'équilibre des forces et des moments de force

$$\sum F : R_A + R_B = P + Q$$

$$\sum M_A : Q(\ell + a) + P \frac{\ell}{2} = R_B \ell$$

D'où

$$R_B = \frac{P}{2} + Q \frac{\ell+a}{\ell}$$

$$R_A = \frac{P}{2} - Q \frac{a}{\ell}$$

2. Energie de déformation

Energie totale accumulée par le système $U = U_1 + U_2$

Energie de déformation du ressort

$$U_1 = \frac{1}{2} \frac{R_B^2}{K}$$

Energie de déformation

$$U_2 = \int_0^{\ell/2} \frac{M_{AD}^2}{2EI} dx + \int_0^{\ell/2} \frac{M_{DB}^2}{2EI} dx' + \int_0^a \frac{M_{BC}^2}{2EI} dx''$$

Hypothèse de Castigliano

$$\delta_C = \left. \frac{\partial U}{\partial Q} \right|_{Q=0} = \left. \frac{\partial U_1}{\partial Q} \right|_{Q=0} + \left. \frac{\partial U_2}{\partial Q} \right|_{Q=0} = 0$$

où $\left. \frac{\partial U_1}{\partial Q} \right|_{Q=0} = \frac{P}{2K} \frac{a+\ell}{\ell}$

et $\left. \frac{\partial U_2}{\partial Q} \right|_{Q=0} = \frac{1}{EI} \left[\int_0^{\ell/2} M_{AD} \frac{\partial M_{AD}}{\partial Q} dx + \int_0^{\ell/2} M_{DB} \frac{\partial M_{DB}}{\partial Q} dx' + \int_0^a M_{BC} \frac{\partial M_{BC}}{\partial Q} dx'' \right]_{Q=0}$

3. Calcul des moments

Sur AD : $0 \leq x \leq \ell/2$

$$M_{AD}(x) = R_A x = \frac{P}{2} x - \frac{aQ}{\ell} x \quad \Rightarrow \quad \frac{\partial M_{AD}}{\partial Q} = -\frac{a}{\ell} x$$

Sur BD : $0 \leq x' \leq \ell/2$

$$M_{BD}(x') = R_B x' - Q(a + x') = \frac{P}{2} x' + Q \left(\frac{a}{\ell} x' - a \right) \quad \Rightarrow \quad \frac{\partial M_{BD}}{\partial Q} = \frac{a}{\ell} x' - a$$

Sur $CB : 0 \leq x'' \leq a$

$$M_{CB}(x'') = -Qx'' \Rightarrow \frac{\partial M_{CB}}{\partial Q} = -x''$$

4. Calcul de la rigidité

$$\frac{\partial U_2}{\partial Q} \Big|_{Q=0} = \frac{1}{EI} \left[\int_0^{\ell/2} -\frac{Pa}{2l} x^2 dx + \int_0^{\ell/2} \left(-\frac{Pa}{2} x' + \frac{Pa}{2l} x'^2 \right) dx' \right] = -\frac{Pa\ell^2}{16EI}$$

$$\frac{\partial U_1}{\partial Q} \Big|_{Q=0} = \frac{P}{2K} \frac{a+\ell}{\ell}$$

La somme des deux expressions permet de trouver la rigidité $\frac{P}{2K} \frac{a+\ell}{\ell} - \frac{Pa\ell^2}{16EI} = 0$

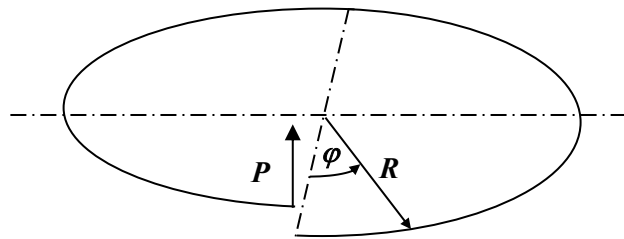
$$K = \frac{8EI}{\ell^3} \left(1 + \frac{\ell}{a} \right)$$

Problème 2

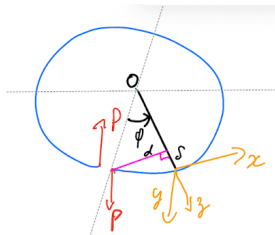
1. Introduction

Castigliano $\delta = \frac{\partial U}{\partial P}$

On ne considère que les moments de flexion et de torsion



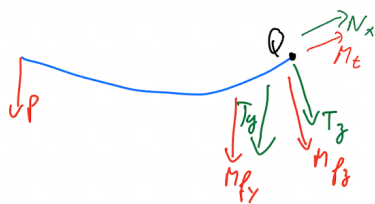
2. Calcul du moment de flexion et de torsion



$$\vec{P} = P \vec{e}_y$$

$$s = R(1 - \cos \varphi)$$

$$d = R \sin \varphi$$



$$\sum F_x = 0 \rightarrow N_x = 0$$

$$\sum F_y = 0 \quad T_y = -P$$

$$\sum F_z = 0 \quad T_z = 0$$

$$\sum M_{x_Q} = 0 \quad M_T - P s = 0$$

$$\sum M_{y_Q} = 0 \quad M_{fy} = 0$$

$$\sum M_{z_Q} = 0 \quad M_{fz} - P d = 0$$

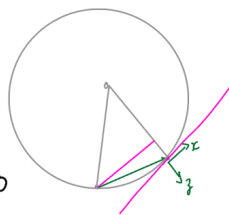
$$M_{fz} = PR \sin \varphi$$

$$M_T = PR \sin \varphi \quad \frac{\partial M_T}{\partial P} = R \sin \varphi$$

$$M_f = PR(1 - \cos \varphi) \quad \frac{\partial M_f}{\partial P} = R(1 - \cos \varphi)$$

$$M_f = PR \sin \varphi \quad \rightarrow \quad \frac{\partial M_f}{\partial P} = R \sin \varphi$$

$$M_t = PR(1 - \cos \varphi) \quad \rightarrow \quad \frac{\partial M_t}{\partial P} = R(1 - \cos \varphi)$$



3. Énergie de déformation

$$U_{M_f} = \int_0^{\ell} \frac{M_f^2}{2EI} ds$$

$$U_{M_{ft}} = \int_0^{\ell} \frac{M_{ft}^2}{2GI_p} ds$$

4. Le déplacement relatif des extrémités est donné par (Castigliano) :

$$\delta = \frac{1}{EI} \int_0^{2\pi} M_f \frac{\partial M_f}{\partial P} ds + \frac{1}{GI_p} \int_0^{2\pi} M_t \frac{\partial M_t}{\partial P} ds \quad \text{avec} \quad ds = R d\phi$$

Donc :

$$\delta = \frac{PR^3}{EI} \int_0^{2\pi} \sin^2 \phi d\phi + \frac{PR^3}{GI_p} \int_0^{2\pi} (1 - \cos \phi)^2 d\phi = PR^3 \left(\frac{\pi}{EI} + \frac{2\pi + \pi}{GI_p} \right)$$

Avec $I_p = 2I$ (section circulaire) :

$$\delta = \frac{\pi PR^3}{EI} \left(1 + \frac{3E}{2G} \right) = 9.2 \text{ mm}$$

Problème 3

1. Introduction

Castigliano

$$\delta_H = \left. \frac{\partial U}{\partial Q} \right|_{Q=0}$$

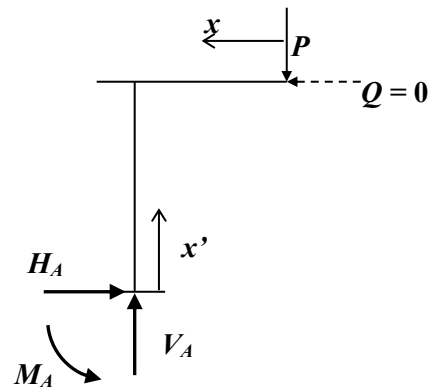
$$\delta_V = \frac{\partial U}{\partial P}$$

Équilibre des forces et des moments

$$H_A = Q$$

$$V_A = P$$

$$M_A = P \ell - Q h$$



2. Partie verticale

$$M_v(x) = -M_A - H_A x = Q(h - x) - P \ell$$

$$\frac{\partial M_v}{\partial P} = -\ell \quad \text{et} \quad \frac{\partial M_v}{\partial Q} = h - x$$

3. Partie horizontale

$$M_h(x) = -P x$$

$$\frac{\partial M_h}{\partial P} = -x \quad \text{et} \quad \frac{\partial M_h}{\partial Q} = 0$$

4. Les déplacements

Déplacement horizontal

$$\delta_H = \left. \frac{\partial U}{\partial Q} \right|_{Q=0} = \frac{1}{EI} \int_0^h M_v(x) \frac{\partial M_v}{\partial Q} \Big|_{Q=0} dx' + 0 =$$

$$\delta_H = \frac{-P \ell h^2}{2EI}$$

Déplacement vertical

$$\delta_V = \frac{\partial U}{\partial P} = \frac{1}{EI} \int_0^\ell M_h(x) \frac{\partial M_h}{\partial P} dx + \frac{1}{EI} \int_0^h M_v(x) \frac{\partial M_v}{\partial P} dx'$$

$$\delta_V = \frac{1}{EI} \int_0^\ell (-P x)(-x) dx + \frac{1}{EI} \int_0^h (-P \ell)(-1) dx'$$

$$\delta_V = \frac{P \ell^2 h}{EI} + \frac{P \ell^3}{3EI}$$

5. Expression du rapport

$$\delta_H = \delta_V$$

$$\frac{P\ell h^2}{2EI} = \frac{P\ell^3}{3EI} + \frac{P\ell^2 h}{EI} \quad \rightarrow \quad 2\ell^2 + 6\ell h - 3h^2 = 0$$

$$2\lambda^2 + 6\lambda - 3 = 0$$

Seule la valeur positive peut être retenue, donc :

$$\lambda = \frac{\sqrt{15} - 3}{2} \approx 0,436$$